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The structure of the Stewartson layers in a gas centrifuge. Part 1. Insulated end plates

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The structure of the Stewartson $E^{\frac{1}{4}}$-layer (E being the Ekman number) in a compressible gas contained in a rapidly rotating cylinder is investigated for the case in which the end plates of the cylinder are thermally insulated. It was found by Matsuda & Hashimoto (1976) that the $E^{\frac{1}{4}}$-layer could not have a relevant structure in the ordinary configuration in which the $E^{\frac{1}{4}}$-layer meets the end plates through its Ekman extension of thickness $E^{\frac{1}{2}}$. In this paper the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region, in addition to the Ekman extension, is considered. The heat generation due to the radial fluid motions in the Ekman extension causes the temperature fields in the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region through which heat is conducted to the side wall of the cylinder. Numerical calculations were made to obtain the temperature fields, which are shown in several figures.

1. Introduction

Compressible fluid motions in a rapidly rotating cylinder have been investigated as a model of gas centrifuges used to enrich uranium in several papers (Sakurai & Matsuda 1974; Nakayama & Usui 1975; Matsuda, Sakurai & Takeda 1975; Matsuda, Hashimoto & Takeda 1976). It was found that the fluid motions strongly depend on the thermal boundary conditions at the cylinder walls, especially those on the end plates (Matsuda & Hashimoto 1976). The effect of the compressibility of the fluid is to produce heat generation or absorption accompanied by radial fluid motion in the radial density stratification. This is dominant in the Ekman layers on the end plates and the Ekman extensions in which the Ekman layers meet the Stewartson layers on the side wall, because the radial motions have greater magnitudes in these regions. When the end plates are thermally conducting, the heat due to the radial motion in the Ekman layer and the Ekman extensions can be removed via the end plates. On the other hand, when the end plates are thermally insulated this is not possible. The present authors (1976) have treated the latter case and shown how the heat produced in the Ekman layer is conducted to the thermally conducting side wall through the inner region and how the axial flow, whose profile plays an essential role in the estimation of the efficiency of uranium enrichment, is affected. However, we failed to give a consistent formulation of the Stewartson $E^{\frac{1}{4}}$-layer and restricted the analysis to the case in which only the Stewartson $E^{\frac{1}{3}}$-layer exists on the side wall, where E is the Ekman number and we assumed $E \ll 1$. The reason for this failure is that the $E^{\frac{1}{4}}$-layer has a simple structure, so that the heat produced in the Ekman extensions of the $E^{\frac{1}{4}}$-layer cannot be conducted to the side wall through the $E^{\frac{1}{4}}$-layer.

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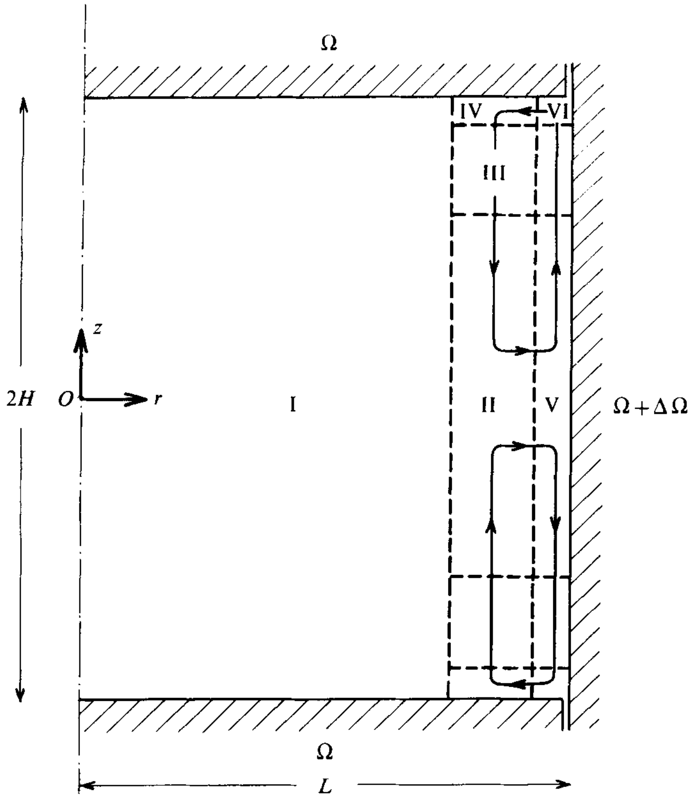


FIGURE 1. The configuration of the system and the structure of the Stewartson layers.

In this paper we study the structure of the $E^{\frac{1}{4}}$-layer when the end plates are thermally insulated and the side wall is kept at a constant temperature. Let us consider the simplest case, in which the existence of the $E^{\frac{1}{4}}$-layer is a necessary condition. The top and bottom end plates of the cylinder of radius L and height $2H$ rotate with the same angular velocity Ω, while the side wall rotates with angular velocity $\Omega+\Delta\Omega$ (see figure 1). Before proceeding with the mathematical formulation of the problem, let us give a brief survey of the results. The $E^{\frac{1}{4}}$-layer in incompressible Boussinesq fluids meets directly its Ekman extensions on the end plates and the temperature and the azimuthal and radial components of the velocity do not depend on the axial position z in the $E^{\frac{1}{4}}$-layer. When the fluid is a compressible gas, the $E^{\frac{1}{4}}$-layer meets its Ekman extensions through square regions of size $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$. In these square regions the above physical quantities depend on z as well as the radial position r. The structure of the Stewartson layers on the side wall is shown schematically in figure 1. Region II is the $E^{\frac{1}{4}}$-layer and region V is the $E^{\frac{1}{3}}$-layer. Regions IV and VI are the Ekman extensions of the $E^{\frac{1}{4}}$-layer and the $E^{\frac{1}{3}}$-layer, respectively. Region III is the square region in question. In this paper we do not treat regions V and VI, in which the higher-order components of the solutions are matched to satisfy the side-wall boundary condition. The details of the $E^{\frac{1}{3}}$-layer and its Ekman extension were investigated by Sakurai & Matsuda (1974).

When the fluid is incompressible, we can obtain consistent solutions without region

III. In the inner region I the fluid rotates rigidly with angular velocity Ω. The difference in azimuthal velocity between region I and the side wall is smoothed through the $E^{\frac{1}{4}}$-layer, region II. The Ekman extensions IV appear on the end plates to match the difference in azimuthal velocity between the $E^{\frac{1}{4}}$-layer and the end plates while the Ekman-layer suction induces the secondary meridional flow in region II. This meridional flow forms circulations from region II to region IV through regions V and VI and then back to region II when $\Delta\Omega > 0$ and vice versa when $\Delta\Omega < 0$ as was shown by Stewartson (1957).

Let us now consider the case of compressible fluids. The basic state is rigid-body rotation of the fluid with angular velocity Ω and constant temperature. The radial density stratification causes heat generation or absorption accompanied by radial fluid motions. In the present case, the radial motions are dominant in regions IV and VI. Because the end plates are thermally insulated, the resultant heat is removed (supplied) to (from) the side wall through region III. Then the temperature field in region III depends on z and r. At the same time a thermal wind appears in this region in the balance between the Coriolis force and the centrifugal buoyancy force.

In §2 the basic equations are given, in §3 the results for regions II, III and IV are obtained and in §4 they are summarized and numerical calculations made to obtain the solutions.

2. Basic equations

We introduce a cylindrical co-ordinate system (r, θ, z) which rotates with angular velocity Ω. The origin of the co-ordinates is the point on the rotation axis midway between the end plates. Consider rigid-body rotation of the fluid with angular velocity Ω and constant temperature T_0 as a basic state. The non-dimensional linearized basic equations governing the deviations in the axisymmetric fluid motion from rigid rotation are

$$\operatorname{div}\mathbf{q} + G_0 ru = 0, \tag{2.1}$$

$$-2v + rT + \frac{1}{G_0}\frac{\partial p}{\partial r} = \frac{E}{\epsilon_R}\left(\mathscr{L}u + \frac{1}{3}\frac{\partial}{\partial r}\operatorname{div}\mathbf{q}\right), \tag{2.2}$$

$$2u = \frac{E}{\epsilon_R}\mathscr{L}v, \tag{2.3}$$

$$\frac{1}{G_0}\frac{\partial p}{\partial z} = \frac{E}{\epsilon_R}\left(\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\operatorname{div}\mathbf{q}\right), \tag{2.4}$$

$$-4hru = (E/\epsilon_R)\,\Delta T, \tag{2.5}$$

where
$$\operatorname{div}\mathbf{q} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \mathscr{L} = \Delta - \frac{1}{r^2}, \tag{2.6}$$

$$G_0 = \frac{ML^2\Omega^2}{RT_0}, \quad E = \frac{\nu}{\Omega L^2}, \quad h = \frac{(\gamma-1)PrG_0}{4\gamma}, \quad \epsilon_R = \exp\left[\frac{G_0}{2}(r^2-1)\right], \tag{2.7}$$

$\mathbf{q}(u, v, w)$ is the velocity vector, M the mean molecular weight of the fluid, R the universal gas constant, ν the kinematic viscosity measured at the side wall and γ the ratio of specific heats. The non-dimensional parameter G_0 is the square of the rotational Mach number, E is the Ekman number, Pr is the Prandtl number and h is a parameter measuring the heat generation or absorption due to the radial fluid motion. In the above expressions the velocity, temperature and position have been non-dimensionalized by $L|\Delta\Omega|$, $G_0 T_0|\Delta\Omega|/\Omega$ and L, respectively. In this paper we treat a rapidly rotating fluid, so that we can neglect the effect of gravity, and we assume that $E \ll 1$ and that G_0 and Pr are of order unity. We also assume that h is $O(E^{\frac{1}{4}})$ since in a uranium gas centrifuge the fluid in question is UF_6, for which γ is 1·067, which is close to unity. The boundary conditions for the present problem are

$$u = w = 0, \quad v = 1, \quad T = 0 \quad \text{on} \quad r = 1 \tag{2.8}$$

and

$$u = v = w = 0, \quad \partial T/\partial z = 0 \quad \text{on} \quad z = \pm A, \tag{2.9}$$

where $A = H/L$ and $\Delta\Omega > 0$.

Let us suppose that the effect of the side wall is confined to the Stewartson layers in the present simple case and that the fluid in the inner region I rotates rigidly with a constant temperature T_i. Denoting the solutions in region I by a suffix i, we have

$$u_i = v_i = w_i = 0, \quad T = \text{constant} = T_i \tag{2.10}$$

and

$$dp_i/dr = -G_0 r T_i. \tag{2.11}$$

The constant T_i is determined after the analysis of the Stewartson layers on the side wall.

3. The analysis of regions II, III and IV

In the present problem, the prescribed values of v and T at the side wall are constant and do not depend on z. This corresponds to the 'symmetric problem' in Hunter's (1967) paper, in which the Stewartson layers in an incompressible fluid were investigated and the inner-region solutions can be matched to their side-wall boundary values without the $E^{\frac{1}{3}}$-layer. We can assume without ambiguity that u, v and T are symmetric functions of z while w is an antisymmetric function of z because the system is symmetric with respect to z. In the following, we restrict our attention to the region $z > 0$.

When the side-wall boundary condition on T is given by a function of z, the constant part of the Fourier series into which the function is expanded in $(-A, A)$ is damped radially in the $E^{\frac{1}{4}}$-layer. Of the remainder, some parts are damped radially in the $E^{\frac{1}{3}}$-layer while others give the boundary condition for the inner-region solution. In this case, therefore, the fluid cannot be in rigid-body rotation with a constant temperature in the inner region (see Matsuda & Hashimoto 1976).

Region II: the $E^{\frac{1}{4}}$-layer

The proper scaling for the variables in the $E^{\frac{1}{4}}$-layer is

$$\left.\begin{aligned} u = E^{\frac{1}{2}}\tilde{u}, \quad v = \tilde{v}, \quad w = E^{\frac{1}{4}}\tilde{w}, \quad p = G_0 p_i + G_0(1+h)E^{\frac{1}{4}}\tilde{p}, \\ T = T_i + h\tilde{T}, \quad \xi = (1-r)E^{-\frac{1}{4}}, \end{aligned}\right\} \tag{3.1}$$

where tildes refer to the $E^{\frac{1}{4}}$-layer and ξ is a stretched radial co-ordinate. Substitution of (3.1) into (2.1)–(2.5) gives us

$$-\partial\tilde{u}/\partial\xi+\partial\tilde{w}/\partial z = 0, \quad (3.2)$$

$$-2\tilde{v}+h\tilde{T}-(1+h)\,\partial\tilde{p}/\partial\xi = 0, \quad (3.3)$$

$$2\tilde{u} = \partial^2\tilde{v}/\partial\xi^2, \quad (3.4)$$

$$\partial\tilde{p}/\partial z = 0, \quad (3.5)$$

$$-4\tilde{u} = \partial^2\tilde{T}/\partial\xi^2. \quad (3.6)$$

Equations (3.2)–(3.6) can be easily integrated and the variables $\tilde{u}, \tilde{v}, \tilde{w}$ and $\tilde{T}$ expressed in terms of $-d\tilde{p}/d\xi$:

$$\tilde{u} = \tfrac{1}{4}\tilde{f}'', \quad \tilde{v} = \tfrac{1}{2}\tilde{f}, \quad \tilde{w} = \tfrac{1}{4}z\tilde{f}''', \quad \tilde{T} = -\tilde{f}, \quad (3.7)$$

where $\tilde{f}(\xi) \equiv -\tilde{p}'$ and the dashes denote differentiation with respect to ξ. The function $\tilde{f}$ must tend to zero as $\xi \to \infty$ because this represents the $E^{\frac{1}{4}}$-layer component of the solution. Now we can determine the values of T_i and $\tilde{f}(0)$. Since $T = T_i+h\tilde{T} = 0$ and $v = \tilde{v} = 1$ at $\xi = 0$, we obtain

$$T_i = 2h, \quad \tilde{f}(0) = 2. \quad (3.8)$$

Region III: the $E^{\frac{1}{3}} \times E^{\frac{1}{4}}$ square region

The variables are scaled according to

$$\left.\begin{aligned} u = E^{\frac{1}{4}}\bar{u}, \quad v = \bar{v}, \quad w = E^{\frac{1}{4}}\bar{w}, \quad p = G_0p_i+G_0(1+h)\,E^{\frac{1}{4}}\bar{p}, \\ T = T_i+\bar{T}, \quad \chi = (A-z)\,E^{-\frac{1}{4}}, \end{aligned}\right\} \quad (3.9)$$

where bars refer to the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region and χ is a stretched axial co-ordinate. We use the same stretched radial co-ordinate ξ as is in the last section. The governing equations for these variables are

$$\partial\bar{w}/\partial\chi = 0, \quad (3.10)$$

$$-2\bar{v}+\bar{T}-(1+h)\,\partial\bar{p}/\partial\xi = 0, \quad (3.11)$$

$$2\bar{u} = \partial^2\bar{v}/\partial\xi^2+\partial^2\bar{v}/\partial\chi^2, \quad (3.12)$$

$$\partial\bar{p}/\partial\chi = 0, \quad (3.13)$$

$$-4h\bar{u} = \partial^2\bar{T}/\partial\xi^2+\partial^2\bar{T}/\partial\chi^2. \quad (3.14)$$

Introducing a function $\phi(\xi,\chi)$ defined by

$$(1+h)\,\phi \equiv 2h\bar{v}+\bar{T}, \quad (3.15)$$

and using (3.12) and (3.14), we see that

$$\partial^2\phi/\partial\xi^2+\partial^2\phi/\partial\chi^2 = 0. \quad (3.16)$$

Then the variables $\bar{u}, \bar{v}$ and $\bar{T}$ can be expressed in terms of $-\bar{p}'$ and ϕ as follows:

$$\bar{u} = \tfrac{1}{4}\bar{f}'', \quad \bar{v} = \tfrac{1}{2}(\phi+\bar{f}), \quad \bar{T} = \phi-h\bar{f}, \quad (3.17)$$

where $\bar{f} \equiv -\bar{p}'$. Matching these solutions to the solutions in the $E^{\frac{1}{4}}$-layer, at $\chi = \infty$, leads us to the relations

$$\bar{w} = \tfrac{1}{4}A\tilde{f}''', \quad \bar{f} = \tilde{f}. \quad (3.18)$$

The function ϕ must tend to zero as $\xi, \chi \to \infty$. As is easily seen from (3.17) and (3.18), $\overline{w}$ and $\overline{u}$ represent the $E^{\frac{1}{4}}$-layer components of the solutions in this square region, while $\overline{v}$ and $\overline{T}$ are composed of the $E^{\frac{1}{4}}$-layer components and the square-region component ϕ. Since the solutions v and T in the $E^{\frac{1}{4}}$-layer already satisfy the side-wall boundary condition, we must have

$$\phi = 0 \quad \text{on} \quad \xi = 0. \tag{3.19}$$

Note that the function ϕ represents the temperature field due to the heat generation or absorption in the Ekman extension of the $E^{\frac{1}{4}}$-layer and the resultant thermal wind and is coupled with the solutions in the Ekman extension by the boundary condition at $\chi = 0$.

Region IV: the Ekman extension

Let us consider a superposition of the components of the solution in the Ekman extension on the components in the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region, which are scaled according to

$$\left.\begin{array}{c} u = \hat{u}, \quad v = \overline{v} + \hat{v}, \quad w = E^{\frac{1}{4}}(\overline{w} + \hat{w}), \quad p = G_0 p_i + G_0(1+h)E^{\frac{1}{4}}(\overline{p} + \hat{p}), \\ T = T_i + \overline{T} + h\hat{T}, \quad \eta = (A - z)E^{-\frac{1}{2}}, \end{array}\right\} \tag{3.20}$$

where carets refer to the Ekman extension and η is a stretched axial co-ordinate. The governing equations are

$$\partial\hat{u}/\partial\xi + \partial\hat{w}/\partial\eta = 0, \tag{3.21}$$

$$-2\hat{v} + h\hat{T} - (1+h)\,\partial\hat{p}/\partial\xi = \partial^2\hat{u}/\partial\eta^2, \tag{3.22}$$

$$2\hat{u} = \partial^2\hat{v}/\partial\eta^2, \tag{3.23}$$

$$\partial\hat{p}/\partial\eta = 0, \quad -4\hat{u} = \partial^2\hat{T}/\partial\eta^2. \qquad (3.24), (3.25)$$

We must choose the solutions of the above equations which tend to zero as $\eta \to \infty$. From (3.23) and (3.25) we have

$$2\hat{v} + \hat{T} = 0. \tag{3.26}$$

Eliminating $\hat{u}, \hat{v}$ and $\hat{p}$ from (3.22), (3.23) and (3.25), we obtain an equation for $\hat{T}$:

$$\partial^4\hat{T}/\partial\eta^4 + 4\sigma^4\hat{T} = 0, \quad \sigma \equiv (1+h)^{\frac{1}{4}}. \tag{3.27}$$

Equation (3.27) is easily integrated subject to the boundary condition

$$\hat{u}(= -\tfrac{1}{4}\partial^2 T/\partial\eta^2) = 0 \quad \text{at} \quad \eta = 0$$

and $\hat{T}$ is given by

$$\hat{T} = C(\xi)\,e^{-\sigma\eta}\cos\sigma\eta. \tag{3.28}$$

It is straightforward to obtain $\hat{u}$, $\hat{v}$ and $\hat{w}$ by using (3.21), (3.25) and (3.26):

$$\hat{u} = -\tfrac{1}{2}\sigma^2 C(\xi)\,e^{-\sigma\eta}\sin\sigma\eta, \tag{3.29}$$

$$\hat{v} = -\tfrac{1}{2}C(\xi)\,e^{-\sigma\eta}\cos\sigma\eta, \tag{3.30}$$

$$\hat{w} = -\tfrac{1}{4}\sigma C'(\xi)\,e^{-\sigma\eta}(\cos\sigma\eta - \sin\sigma\eta). \tag{3.31}$$

The boundary conditions at the top end plate,

$$\left.\begin{array}{l} v = \overline{v} + \hat{v} = 0, \quad w = E^{\frac{1}{4}}(\overline{w} + \hat{w}) = 0 \\ \dfrac{\partial T}{\partial z} = \dfrac{\partial \overline{T}}{\partial z} + h\dfrac{\partial \hat{T}}{\partial z} = E^{-\frac{1}{4}}\dfrac{\partial \overline{T}}{\partial \chi} + E^{-\frac{1}{2}}h\dfrac{\partial \hat{T}}{\partial \eta} = 0 \end{array}\right\} \quad \text{on} \quad \eta = \chi = 0, \qquad (3.32), (3.33)$$

and (3.34)

can be expressed in terms of ϕ, $\tilde{f}$ and C as

$$\left.\begin{array}{l}\phi+\tilde{f}-C=0, \quad A\tilde{f}''''-\sigma C'=0 \\ \partial\phi/\partial\chi-\sigma\beta_0 C=0\end{array}\right\} \text{ on } \chi=0, \qquad (3.35),(3.36),(3.37)$$

and

where $\beta_0 \equiv hE^{-\frac{1}{4}}$ and use has been made of $\tilde{f}=\bar{f}$. On eliminating C from (3.35)–(3.37), we obtain an ordinary differential equation for $\tilde{f}(\xi)$ and the boundary condition for $\phi(\xi,\chi)$ at $\chi=0$:

$$\tilde{f}''-\frac{\sigma}{A}\tilde{f}=\frac{\sigma}{A}\phi(\xi,0) \qquad (3.38)$$

and

$$\partial\phi/\partial\chi-\sigma\beta_0\phi=\sigma\beta_0\tilde{f} \quad \text{on} \quad \chi=0, \qquad (3.39)$$

respectively.

Before closing this section, we must mention that we cannot have a relevant formulation of the $E^{\frac{1}{4}}$-layer without the $E^{\frac{1}{2}}\times E^{\frac{1}{2}}$ square region. If there are no square-region components in the solution, $C(\xi)$ must vanish from (3.28) to satisfy the thermal boundary condition $\partial\hat{T}/\partial\eta=0$ at the end plates because the $E^{\frac{1}{4}}$-layer component of the temperature does not depend on z. This means that the Ekman extension does not exist. Thus the $E^{\frac{1}{4}}$-layer components of the solution must satisfy the boundary conditions on the end plates themselves, which is impossible as may be seen from (3.7). This is the discrepancy which the present authors met in their previous paper (1976).

4. Summary and numerical results

Let us summarize the results obtained in the last section. The equations for $\phi(\xi,\chi)$ and $\tilde{f}(\xi)$ are

$$\partial^2\phi/\partial\xi^2+\partial^2\phi/\partial\chi^2=0, \qquad (4.1)$$

$$\frac{d^2\tilde{f}}{d\xi^2}-\frac{\sigma}{A}\tilde{f}=\frac{\sigma}{A}\phi(\xi,0). \qquad (4.2)$$

The boundary conditions are

$$\partial\phi/\partial\chi-\sigma\beta_0\phi=\sigma\beta_0\tilde{f} \quad \text{on} \quad \chi=0, \qquad (4.3)$$

$$\phi=0, \quad \tilde{f}=2 \quad \text{on} \quad \xi=0, \qquad (4.4)$$

and

$$\phi,\tilde{f}\rightarrow 0 \quad \text{as} \quad \xi\rightarrow\infty; \quad \phi\rightarrow 0 \quad \text{as} \quad \chi\rightarrow\infty. \qquad (4.5)$$

When h is of order $E^{\frac{1}{4}}$, which has been assumed implicitly in the scaling in the last section, β_0 is of order unity and we introduce Fourier transforms to obtain the solutions of (4.1) and (4.2) subject to (4.3)–(4.5). We denote the Fourier sine transforms of ϕ and $\tilde{f}$, with respect to ξ, by Φ and F:

$$\Phi(\lambda,\chi)=\int_0^\infty \phi(\xi,\chi)\sin(\xi\lambda)\,d\xi, \qquad (4.6)$$

$$F(\lambda)=\int_0^\infty \tilde{f}(\xi)\sin(\xi\lambda)\,d\xi. \qquad (4.7)$$

Then, from the Fourier sine transforms of (4.1), we obtain

$$\partial^2\Phi/\partial\chi^2 - \lambda^2\Phi = 0. \tag{4.8}$$

The relevant solution of (4.8) is

$$\Phi = D(\lambda)\, e^{-\lambda\chi}. \tag{4.9}$$

The Fourier sine transforms of (4.2) and (4.3) are respectively

$$\left(\lambda^2 + \frac{\sigma}{A}\right) F = \lambda \tilde{f}(0) - \frac{\sigma}{A}\,\Phi(\lambda, 0) \tag{4.10}$$

and

$$[\partial\Phi/\partial\chi]_{\chi=0} - \sigma\beta_0\,\Phi(\lambda,0) = \sigma\beta_0 F. \tag{4.11}$$

Substitution of (4.9) into (4.10) and (4.11) and use of $\tilde{f}(0) = 2$ gives us

$$F = 2(\lambda + \sigma\beta_0)/(\lambda^2 + \sigma\beta_0\lambda + \sigma/A), \tag{4.12}$$

$$D = -2\sigma\beta_0/(\lambda^2 + \sigma\beta_0\lambda + \sigma/A). \tag{4.13}$$

By inversion of (4.9) and (4.12), ϕ and $\tilde{f}$ can be expressed as

$$\phi(\xi, \chi) = \frac{2}{\pi}\int_0^\infty \Phi(\lambda, \chi)\sin(\lambda\xi)\, d\lambda, \tag{4.14}$$

$$\tilde{f}(\xi) = \frac{2}{\pi}\int_0^\infty F(\lambda)\sin(\lambda\xi)\, d\lambda. \tag{4.15}$$

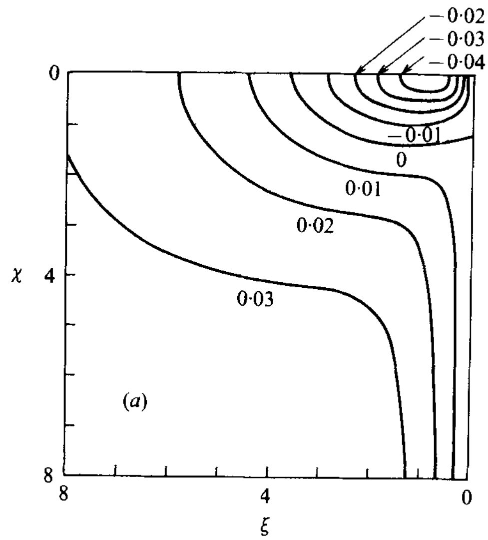
The integrals in (4.14) and (4.15) were calculated numerically. Figures 2(*a*), (*b*) and (*c*) show the temperature field in the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region and the Ekman extension in the upper corner of the cylinder for the three cases (*a*) $h = 0{\cdot}05$, (*b*) $h = 0{\cdot}2$ and (*c*) $h = 0{\cdot}5$ with $E^{\frac{1}{4}} = 0{\cdot}2$ and $A = 1$. In these figures the upper end plate corresponds to $\chi = 0$ and the side wall to $\xi = 0$. The region near $\chi = 0$ is the Ekman extension. As is discussed in §1, there is radially inward ($+\xi$ direction) motion of the fluids, so that a fluid element cools in the radial pressure stratification due to the rigid-body rotation. Because the end plates are thermally insulated, the heat must be supplied from the side wall to maintain the fluid motion. The strong heat flux from the side wall to the Ekman extension can be observed in figures 2(*a*), (*b*) and (*c*).

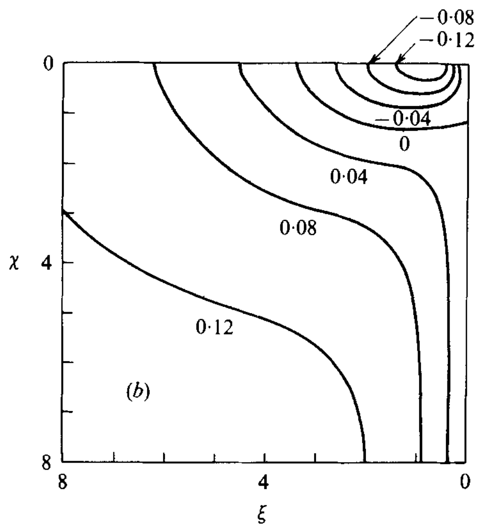
When h is less than $O(E^{\frac{1}{4}})$ [in Matsuda & Hashimoto's (1976) paper h was estimated to be $O(E^{\frac{1}{2}})$], we see from (4.3) that $\partial\phi/\partial\chi = 0$ at $\chi = 0$. In that case the relevant solution of (4.1) is apparently $\phi = 0$. Then we obtain from (4.2)

$$\tilde{f} = 2\exp\{-(\sigma/A)^{\frac{1}{2}}\,\xi\}. \tag{4.16}$$

This is the ordinary $E^{\frac{1}{4}}$-layer solution. A non-zero component of ϕ appears at a reduced order and this couples the solution with the Ekman-extension solution, as has already been shown in (4.16) and (3.34).

When h is greater than $O(E^{\frac{1}{4}})$, (4.3) leads to $\phi + \tilde{f} = 0$ at $\chi = 0$. Then, from (3.35), we observe that $C(\xi) = 0$, which means that the Ekman extensions vanish. This is because the heat absorption accompanied by radial fluid motion in the Ekman extensions is strong in this case while the heat flux from the side wall to the Ekman extension via the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region is not sufficiently large to maintain the fluid motion in the Ekman extension under the scaling in §3, so that the Ekman extension itself must vanish. In addition, from (4.2) we see that $\tilde{f}'' = 0$ and we cannot have the relevant





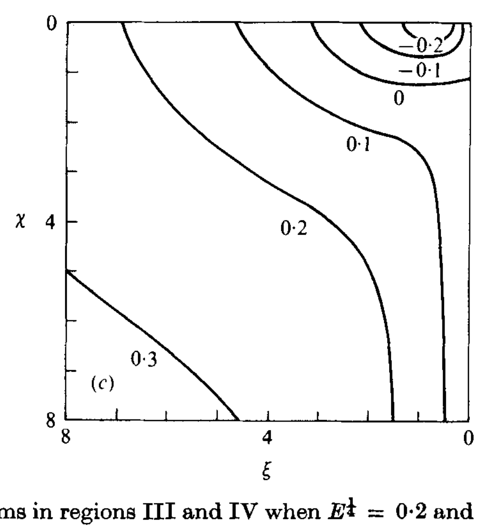


FIGURE 2. Isotherms in regions III and IV when $E^{\frac{1}{4}} = 0{\cdot}2$ and $A = 1$. (*a*) $h = 0{\cdot}05$. (*b*) $h = 0{\cdot}2$. (*c*) $h = 0{\cdot}5$.

solution for $\tilde{f}$ which satisfies the conditions $\tilde{f}(0) = 2$ and $\tilde{f}(\infty) = 0$. In the case $h \gg E^{\frac{1}{4}}$ we fail to give an appropriate formulation of the $E^{\frac{1}{4}}$-layer even if we consider the $E^{\frac{1}{4}} \times E^{\frac{1}{4}}$ square region.

Finally, we must mention that the meridional secondary flow in the $E^{\frac{1}{4}}$-layer does not vanish at $\xi = 0$. As was discussed in § 1, this meridional secondary flow is matched to its side-wall boundary condition via the $E^{\frac{1}{3}}$-layer. An analysis of the $E^{\frac{1}{3}}$-layer was given by Sakurai & Matsuda (1974) for a rapidly rotating compressible fluid and we do not treat it in this paper.

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